THE CONNECTION BETWEEN THE SECOND ORDER HOLOMORPHIC JETS BUNDLE AND THE ECONOMETRIES

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Abstract

We study the econometric models which can take the form of geometrical objects known as manifolds. From this point of view, we analyze the holomorphic jets bundle of order two $J^{(2,0)}M$. We obtain the geodesic curves of the energy, using the first variation of the energy for the curves from $J^{(2,0)}M$, depending on the general metric structure G and a fixed N-linear connection D.

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1. INTRODUCTION

The differential geometry and statistical modelling are compatible in a fundamental sense: in many situations, it is required that statistical inferences do not depend on the way that the statistical model has been parameterized, while one definition of geometry is the study of those things which are invariant under a change of coordinates.

Many statistical procedures have very natural geometric interpretation, for examples: regression or dimension reduction of a statistic.

The statistical examples to which the geometric theory most naturally applies is the class of *full* and *curved exponential families.*

The study of differential geometry in statistics has led to important advances in a variety of fields, including the development of new geometries for statistics, higher order asymptotic theory and curved exponential families, [2].

The theory of manifolds is fundamental to the development of differential geometry. In geometry, the shortest path between two points is a straight line and the lenght of this line defines the distance between these points. This use of minimum path lengths provided the intuitive basis for the metric axioms: nonnegativity, symmetry and the triangle inequality. Metrics are related to metric tensors on manifolds and such tensors define geodesic distances.

Let M be a complex manifold, $dim_C M = n$, (z^i) be the complex coordinates in a local chart. The complexified tangent bundle $T_C M$ admits the classical decomposition $T_C M = T'M \oplus T''M$, where $T'M$ is a holomorphic vector bundle over M and its conjugate $T''M$ is the anti-holomorphic tangent bundle.

The holomorphic bundle of k—th *order jets differential* was introduced by Green and Griffiths in [1, 3] as the classes of sheaf of germs of holomorphic curves $\{f : \Delta_r \to M, f \in H_{z0}, f(0) = z_0\}$ depending on a complex parameter θ, whose partial derivatives up to order k coincide. A detailed construction of $\pi^{(k,0)} : J^{(k,0)}M \to M$ fibre bundle structure, was discussed in [4]. $J^{(k,0)}M$ has a structure of complex differentiable manifold, whose geometry will be briefly recalled bellow for $k = 2$.

On the complex manifold $J^{(2,0)}M$, in a local chart, the coordinates are denoted by $Z = (z^k, \eta^k, \zeta^k)$, $k = \overline{1,n}$, and their changes are according to the following rules:

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$$z'^i = z'^i(z);$$
$$\eta'^i = \frac{\partial z'^i}{\partial z^j}\eta^j; \qquad (1.1)$$
$$2\zeta'^i = \frac{\partial \eta'^i}{\partial z^j}\eta^j + 2\frac{\partial \eta'^i}{\partial \eta^j}\zeta^j.$$

A local basis in the holomorphic bundle $T'(J^{(2,0)}M)$ is $\left\{\frac{\partial}{\partial z^i}, \frac{\partial}{\partial \eta^i}, \frac{\partial}{\partial \zeta^i}\right\}$ and the corresponding basis in $T''(J^{(2,0)}M)$ is obtained by conjugation everywhere. The changes of the local basis are made according with the following rules:

$$\frac{\partial}{\partial z^j} = \frac{\partial z'^i}{\partial z^j}\frac{\partial}{\partial z'^i} + \frac{\partial \eta'^i}{\partial z^j}\frac{\partial}{\partial \eta'^i} + \frac{\partial \zeta'^i}{\partial z^j}\frac{\partial}{\partial \zeta'^i}; \qquad (1.2)$$

$$\frac{\partial}{\partial \eta^j} = \frac{\partial \eta'^i}{\partial \eta^j}\frac{\partial}{\partial \eta'^i} + \frac{\partial \zeta'^i}{\partial \eta^j}\frac{\partial}{\partial \zeta'^i};$$

$$\frac{\partial}{\partial \zeta^j} = \frac{\partial \zeta'^i}{\partial \zeta^j}\frac{\partial}{\partial \zeta'^i}.$$

By conjugation everywhere in (1.2), we obtain the corresponding conjugate basis from $T''_z(J^{(2,0)}M)$.

A complex nonlinear connection, (c.n.c.) in brief, is given by a distribution $H(J^{(2,0)}M)$ at every point $Z \in J^{(2,0)}M$ which is supplementary to $V_1(J^{(2,0)}M)$ in $T'(J^{(2,0)}M)$, where $V_{1Z}(J^{(2,0)}M)$ is spanned by $\left\{\frac{\partial}{\partial \eta^j}, \frac{\partial}{\partial \zeta^j}\right\}$, in a local chart. With $V_2(J^{(2,0)}M)$ we denote the vertical bundle and locally it is spanned in Z by $\left\{\frac{\partial}{\partial \zeta^j}\right\}$. By conjugation, we obtain the decomposition for $T_C(J^{(2,0)}M)$. A local basis in $H_Z(J^{(2,0)}M)$ is $\frac{\delta}{\delta z^j} = \frac{\partial}{\partial z^j} - \overset{(1)}{\widehat{N_j^i}}\frac{\partial}{\partial \eta^i} - \overset{(2)}{\widehat{N_j^i}}\frac{\partial}{\partial \zeta^i}$ and it is called *the adapted basis* of the (c.n.c.) iff $\frac{\delta}{\delta z^j} = \frac{\partial z'^i}{\partial z^j}\frac{\delta}{\delta z'^i}$. If F is the natural almost tangent structure on $J^{(2,0)}M$, given by $F\left(\frac{\partial}{\partial z^j}\right) = \frac{\partial}{\partial \eta^j}$; $F\left(\frac{\partial}{\partial \eta^j}\right) = \frac{\partial}{\partial \zeta^j}$; $\left(\frac{\partial}{\partial \zeta^j}\right) = 0$,which sends $H(J^{(2,0)}M)$ into $V_1(J^{(2,0)}M)$ and this into $V_2(J^{(2,0)}M) = \ker F$, then $F\left(\frac{\delta}{\delta z^j}\right) =: \frac{\delta}{\delta \eta^j} = \frac{\partial}{\partial \eta^j} - \overset{(1)}{\widehat{N_j^i}}\frac{\partial}{\partial \zeta^i}$ span a local adapted basis in $V_{1Z}(J^{(2,0)}M)$.

The changes (1.1) of coordinates on $J^{(2,0)}M$ produce the changes of the coefficients $\overset{(1)}{\widehat{N_i^j}}$ and $\overset{(2)}{\widehat{N_i^j}}$ of the (c.n.c.) in the form:

$$\overset{(1)}{\widehat{N'^i_k}}\frac{\partial z'^k}{\partial z^j} = \frac{\partial z'^i}{\partial z^k}\overset{(1)}{\widehat{N_j^k}} - \frac{\partial \eta'^i}{\partial z^j}; \qquad (1.2)$$
$$\overset{(2)}{\widehat{N'^i_k}}\frac{\partial z'^k}{\partial z^j} = \frac{\partial z'^i}{\partial z^k}\overset{(2)}{\widehat{N_j^k}} + \frac{\partial \eta'^i}{\partial z^k}\overset{(1)}{\widehat{N_j^k}} - \frac{\partial \zeta'^i}{\partial z^j};$$

The adapted basis will change as follows: $\frac{\delta}{\delta z^j} = \frac{\partial z'^i}{\partial z^j}\frac{\delta}{\delta z'^i}$, $\frac{\delta}{\delta \eta^j} = \frac{\partial z'^i}{\partial z^j}\frac{\delta}{\delta \eta'^i}$ and obviously $\frac{\delta}{\delta \zeta^j} = \frac{\partial z'^i}{\partial z^j}\frac{\delta}{\delta \zeta'^i}$ if we denote by $\frac{\delta}{\delta \zeta^j} := \frac{\partial}{\partial \zeta^j}$, so these fields are changing like vectors on the base manifold ***M.*** Further on, we will use for the adapted basis $\left\{\frac{\delta}{\delta z^i},\frac{\delta}{\delta\eta^i},\frac{\partial}{\partial\zeta^i}\right\}$ the abreviations $\{\delta_{0i},\delta_{1i},\delta_{2i}\}$. The adapted basis on T"($J^{(2,0)}$M) is obtained by conjugation. Let $\{dzi,\delta\eta^i,\delta\zeta^i\}_{\ i=\overline{1,n}}$ be the dual of the obtained adapted basis. If $\delta\eta^i = d\eta^i + \overset{(1)}{\widetilde{M_i^j}}\, dz^j$ and $\delta\zeta^i = d\zeta^i + \overset{(1)}{\widetilde{M_i^j}}\, d\eta^j + \overset{(2)}{\widetilde{M_i^j}}\, dz^j$, then $\overset{(1)}{\widetilde{M_i^j}} = \overset{(1)}{\widetilde{N_i^j}}$ and $\overset{(2)}{\widetilde{M_j^i}} = \overset{(2)}{\widetilde{N_j^i}} + \overset{(1)}{\widetilde{N_k^i}}\,\overset{(1)}{\widetilde{N_j^k}}$. For details see [4].

A metric structure on $J^{(2,0)}M$ is given, in general form, by

$$G = \overset{(0)}{\widetilde{g_{i\bar{j}}}}\ dzi \otimes d\bar{z}^j + \overset{(1)}{\widetilde{g_{i\bar{j}}}}\ \delta\eta^i \otimes \delta\bar{\eta}^j + \overset{(2)}{\widetilde{g_{i\bar{j}}}}\ \delta\zeta^i \otimes \delta\bar{\zeta}^j \tag{1.4}$$

where $\overset{(1)}{\widetilde{g_{i\bar{j}}}}$ are d—metric Hermitian tensors on $J^{(2,0)}M$. In particular $\overset{(1)}{\widetilde{g_{i\bar{j}}}} = g_{i\bar{j}}$, where $g_{i\bar{j}}$ could be derived from a complex Lagrangian function $L : J^{(2,0)}M \to R$ by $g_{i\bar{j}} = \frac{\partial^2 L}{\partial\zeta^i\partial\bar{\zeta}^j}$ and then the space *(M,L)* is called a *second order complex Lagrange space,* [5]. If $L(z, \lambda\eta, \lambda^2\zeta) = |\lambda|^4 L(z, \eta, \zeta)$, $\forall\lambda\in\mathbf{C}$, then we say to be a *second order complex Finsler space,* and then $g_{i\bar{j}}(z, \lambda\eta, \lambda^2\zeta) = g_{i\bar{j}}(z, \eta, \zeta)$ is (0,0) homogeneous, [6].

Finally, we recall that in [4] a special derivative law on $J^{(2,0)}M$ was introduced, namely the *normal complex linear connection, N*-(c.l.c.), which preserves the distributions and has some special properties. A *N*-(c.l.c.) is well given in an adapted frame by a set of coefficients $D\Gamma = (L^i_{jk}, L^{\bar{i}}_{\bar{j}k}, F^i_{jk}, F^{\bar{i}}_{\bar{j}k}, C^i_{jk}, C^{\bar{i}}_{\bar{j}k})$, which are changed at (1.1) as follows:

$$L'^i_{jk} = \frac{\partial z'^i}{\partial z^r}\frac{\partial z^p}{\partial z'^j}\frac{\partial z^q}{\partial z'^k}L^r_{pq} + \frac{\partial z'^i}{\partial z^p}\frac{\partial^2 z^p}{\partial z'^j z'^k}\ ; \tag{1.5}$$

and all the others are d−tensors, that is $F'^i_{jk} = \frac{\partial z'^i}{\partial z^r}\frac{\partial z^p}{\partial z'^j}\frac{\partial z^q}{\partial z'^k}F^r_{pq}$ and similar for C^i_{jk}.

Locally, for α = 1,2,3 we have: $D_{\delta_{0k}}\delta_{\alpha j} = L^i_{jk}\delta_{\alpha i}$; $D_{\delta_{0k}}\delta_{\alpha\bar{j}} = L^{\bar{i}}_{\bar{j}k}\delta_{\alpha\bar{i}}$; $D_{\delta_{1k}}\delta_{\alpha j} = F^i_{jk}\delta_{\alpha i}$; $D_{\delta_{1k}}\delta_{\alpha\bar{j}} = F^{\bar{i}}_{\bar{j}k}\delta_{\alpha\bar{i}}$; $D_{\delta_{2k}}\delta_{\alpha j} = C^i_{jk}\delta_{\alpha i}$; $D_{\delta_{2k}}\delta_{\alpha\bar{j}} = C^{\bar{i}}_{\bar{j}k}\delta_{\alpha\bar{i}}$.

In a second order complex Lagrange space there exists a special complex nonlinear connection named the Chern-Lagrange (c.n.c.), [5] , where

$$\overset{(1)CL}{\widetilde{M_i^j}} = g^{\bar{m}i}\frac{\partial^2 L}{\partial\eta^j\partial\bar{\zeta}^m};\quad \overset{(2)CL}{\widetilde{M_i^j}} = g^{\bar{m}i}\frac{\partial^2 L}{\partial z^j\partial\bar{\zeta}^m} \tag{1.6}$$

and with respect to its adapted frame the following connection named the *Chern-Lagrange complex linear connection,* given by

$$L^i_{jk} = g^{\bar{m}i}\frac{\delta g_{j\bar{m}}}{\delta z^k};\ F^i_{jk} = g^{\bar{m}i}\frac{\delta g_{j\bar{m}}}{\delta\eta^k};\ C^i_{jk} = g^{\bar{m}i}\frac{\delta g_{j\bar{m}}}{\delta\zeta^k} \tag{1.7}$$

and $L^{\bar{i}}_{\bar{j}k} = F^{\bar{i}}_{\bar{j}k} = C^{\bar{i}}_{\bar{j}k} = 0$, i.e *D* is of (1,0)-type, is a normal complex linear connection. Moreover, the Chern-Lagrange (c.l.c.) is one metrical, i.e. DG = 0.

2. GEODESICS ON $J^{(2,0)}M$

Let $\boldsymbol{\sigma}: [\boldsymbol{a}, \boldsymbol{b}] \to \boldsymbol{M}, \boldsymbol{\sigma}(\boldsymbol{t}) = \left(\boldsymbol{z^k}(\boldsymbol{t})\right)$, be a curve on *M*. A curve on $J^{(2,0)}M$ is a differentiable map $\boldsymbol{c}: [\boldsymbol{a}, \boldsymbol{b}] \to \boldsymbol{J^{(2,0)}M}$ given by $\boldsymbol{t} \to \left(\boldsymbol{z^k}(\boldsymbol{t}), \boldsymbol{\eta^k}(\boldsymbol{t}), \boldsymbol{\zeta^k}(\boldsymbol{t})\right)$, where $\boldsymbol{\eta^k}(\boldsymbol{t}) = \frac{\boldsymbol{dz^k}}{\boldsymbol{dt}}$ and $\boldsymbol{\zeta^k}(\boldsymbol{t}) = \frac{\boldsymbol{1}}{\boldsymbol{2}}\frac{\boldsymbol{d^2z^k}}{\boldsymbol{dt^2}}$. We denote by $\boldsymbol{t} \to \left(\boldsymbol{z^k}(\boldsymbol{t}), \boldsymbol{\eta^k}(\boldsymbol{t})\right)$ a curve in the holomorphic fibre bundle $\boldsymbol{\pi}: \boldsymbol{J^{(2,0)}M} \to \boldsymbol{T'M}$, which is a base manifold for $\boldsymbol{\pi_1}: \boldsymbol{J^{(2,0)}M} \to \boldsymbol{T'M}$.

The tangent vectors field at the curve *c* is given by $\boldsymbol{V} \coloneqq \frac{\boldsymbol{dc}}{\boldsymbol{dt}}$, which, in the adapted frame of a (c.n.c.), is written as

$$\frac{\boldsymbol{dc}}{\boldsymbol{dt}} \coloneqq \dot{c}_t + \overline{\dot{c}_t} = \frac{dz^k}{dt}\delta_{0k} + \frac{\delta\eta^k}{dt}\delta_{1k} + \frac{\delta\zeta^k}{dt}\delta_{2k} + \frac{d\bar{z}^k}{dt}\delta_{0\bar{k}} + \frac{\delta\bar{\eta}^k}{dt}\delta_{1\bar{k}} + \frac{\delta\bar{\zeta}^k}{dt}\delta_{2\bar{k}} \,. \tag{2.1}$$

It results that $V = \overset{(\alpha)}{\widehat{V^k}}\,\delta_{\alpha k} + \overset{(\alpha)}{\overline{\widehat{V^k}}}\,\delta_{\alpha\bar{k}}$, where $\overset{(0)}{\widehat{V^k}} = \frac{dz^k}{dt}$, $\overset{(1)}{\widehat{V^k}} = \frac{\delta\eta^k}{dt}$, $\overset{(2)}{\widehat{V^k}} = \frac{\delta\zeta^k}{dt}$ are the derivations with respect to $t \in \boldsymbol{R}$ along the curve *c* of the (c.n.c.) dual basis, $\overset{(\alpha)}{\overline{\widehat{V^k}}}$ are obtained by conjugation in $\overset{(\alpha)}{\widehat{V^k}}$.

Obviously, V is global defined on $T_C(J^{(2,0)}M)$ along the curve *c*. We can define the $h-, v_1-, v_2-$curves on $J^{(2,0)}M$, depending on the vanishing of the $\overset{(\alpha)}{\widehat{V^k}}$.

For example, a *h*−curve is defined by $\overset{(1)}{\widehat{V^k}} = \overset{(2)}{\widehat{V^k}} = 0$. In this section, the problem is the first variation of the curve *c*, in the general case for a fixed *N*−(c.l.c.) *D* and a metric *G* given by (1.4). The method is similarly with the one from the real case for the fibre bundle Osc^2M, see [9, 4].

Let $\sigma : t \to (zk(t))$ be a fixed curve on *M* and *c* its extension at $J^{(2,0)}M$. We consider $\Sigma: [-\varepsilon, \varepsilon] \times [a, b] \to M$, a variation $(s, t) \to \Sigma^k(s, t) \equiv z^k(s, t)$ of the curve σ with $\Sigma(0, t) = \sigma(t)$. For a fixed s, the above curves Σ define the curves $\tilde{c}(s, t)$ on $J^{(2,0)}M$. We have the following diagram:

$$\begin{array}{ccc} \gamma^*(J^{(2,0)}M) & \overset{\Gamma}{\to} & J^{(2,0)}M \\ p_1 \downarrow & & \downarrow \pi_1 \\ \Sigma^*(T'M) & \overset{\gamma}{\to} & T'M \\ p \downarrow & & \downarrow \pi \\ [-\varepsilon, \varepsilon] \times [a, b] & \overset{\Sigma}{\to} & M \end{array} \tag{2.2}$$

where $\Sigma^*(T'M)$ and $\gamma^*(J^{(2,0)}M)$ are the pull-back bundles of the $T'M$ and $J^{(2,0)}M$ using the maps $\boldsymbol{\Gamma}$ and $\boldsymbol{\gamma}$, respectively. The maps $\boldsymbol{\gamma}$ and $\boldsymbol{\Gamma}$ are $\gamma: (s, t) \to (z^k(s, t), \eta^k(s, t))$ and $\Gamma: (s, t) \to \left(z^k(s, t), \eta^k(s, t), \zeta^k(s, t)\right)$.

We consider the tangent vectors $V = \overset{(\alpha)}{\widehat{V^k}}\,\delta_{\alpha k} + \overset{(\alpha)}{\overline{\widehat{V^k}}}\,\delta_{\alpha\bar{k}} = \dot{c}_t(s) + \overline{\dot{c}_t}(s)$, which for *s* is a parameter, and respectively the deviation vectors $U = \overset{(\alpha)}{\widehat{U^k}}\,\delta_{\alpha k} + \overset{(\alpha)}{\overline{\widehat{U^k}}}\,\delta_{\alpha\bar{k}} = \dot{c}_s(t) + \overline{\dot{c}_s}(t)$, where $\overset{(0)}{\widehat{U^k}} = \frac{dz^k}{ds}$, $\overset{(1)}{\widehat{U^k}} = \frac{\delta\eta^k}{ds}$, $\overset{(2)}{\widehat{U^k}} = \frac{\delta\zeta^k}{ds}$. The variation Σ is with fixed endpoints, if $\tilde{c}(s,a) = c(a)$, $\tilde{c}(s,b) = c(b)$ and $V(a) = V(b) = 0$.

Let Ω_c be the variations set of the curve c, $L: \Omega_c \to \boldsymbol{R}$ a differentiable map and $L_*: T\Omega_c \to T\boldsymbol{R}_{L(c)}$ its tangent map with $L_*(U) := \frac{dL(\tilde{c})}{ds}|_{s=0} \frac{d}{dt}|_{L(\tilde{c})}$.

As in the classic variational theory, we name the *extremal value* of L a curve $\tilde{c} \in \Omega_c$ with $L_*(U) = 0$, i.e. $\frac{dL(\tilde{c})}{ds}|_{s=0} = 0$.

Further on, let us see in which circumstances we can define L. For the fixed N-(c.l.c.) D and the metric G given by (1.4), with DG=0, we define the *energy* along the curve $c: t \to \left(z^k(s,t), \eta^k(s,t), \zeta^k(s,t)\right)$ with fixed endpoints, being

$$E(c) = \frac{1}{2}\int_a^b G(\dot{c}_t, \bar{\dot{c}}_t)dt. \tag{2.3}$$

Let's determine the extremal values of the energy function, i.e.

$$\frac{1}{2}\frac{dE(\tilde{c})(s,t)}{ds}|_{s=0} = \frac{1}{2}\int_a^b \frac{dG(\dot{c}_t, \bar{\dot{c}}_t)}{ds}dt = 0. \tag{2.4}$$

Taking into account that $G(\dot{c}_t, \bar{\dot{c}}_t) = \Sigma_{\alpha=0,1,2} G(\overset{(\alpha)}{\widehat{V}}, \overset{(\alpha)}{\bar{\widehat{V}}})$ and $\frac{d}{ds} = \frac{dz^k}{ds}\delta_{0k} + \frac{\delta\eta^k}{ds}\delta_{1k} + \frac{\delta\zeta^k}{ds}\delta_{2k} + \frac{d\bar{z}^k}{ds}\delta_{0\bar{k}} + \frac{\delta\bar{\eta}^k}{ds}\delta_{1\bar{k}} + \frac{\delta\bar{\zeta}^k}{ds}\delta_{2\bar{k}} = U + \bar{U} := \dot{c}_s(t) + \bar{\dot{c}}_s(t)$, we obtain

$$\frac{1}{2}\frac{dG(\dot{c}_t, \bar{\dot{c}}_t)}{ds}|_{s=0} = \frac{1}{2}\Sigma_{\alpha=0,1,2}(\overset{(\alpha)}{\widehat{U}}, \overset{(\alpha)}{\bar{\widehat{U}}})G(\overset{(\alpha)}{\widehat{V}}, \overset{(\alpha)}{\bar{\widehat{V}}}).$$

Moreover, D is metric, i.e. $XG(Y,Z) = G(D_XY,Z) + G(Y,D_XZ)$ and Hermitian, $G(X,\bar{Y}) = \overline{G(Y,\bar{X})}$, and by expanding the calculation from the last formua, we obtain:

$$\frac{1}{2}\frac{dG(\dot{c}_t, \bar{\dot{c}}_t)}{ds} = Re\{G\left(D_{\dot{c}_s}\dot{c}_t, \bar{\dot{c}}_t\right) + G\left(D_{\bar{\dot{c}}_s}\dot{c}_t, \bar{\dot{c}}_t\right)\}$$

$$= Re\left\{G\left(\boldsymbol{T}(\dot{c}_s, \dot{c}_t) + D_{\bar{\dot{c}}_t}\dot{c}_t, \bar{\dot{c}}_t\right) + G(\boldsymbol{T}(\bar{\dot{c}}_s, \dot{c}_t) + D_{\bar{\dot{c}}_t}\dot{c}_t, \bar{\dot{c}}_t)\right\},$$

where **T**(X,Y) is the connection torsion D, written in the adapted frames given by the vectors $\dot{c}_s, \dot{c}_t$ and their conjugates.

So,

$$\frac{1}{2}\frac{dE(\tilde{c})(s,t)}{ds}|_{s=0} = \int_a^b Re\{G(\boldsymbol{T}(\dot{c}_s + \bar{\dot{c}}_s, \dot{c}_t), \bar{\dot{c}}_t) + G\left(D_{\dot{c}_t}\dot{c}_t, \bar{\dot{c}}_t\right) + G(D_{\bar{\dot{c}}_t}\dot{c}_t, \bar{\dot{c}}_t)\}dt.$$

Integrating by parts, we obtain that

$$\frac{1}{2}\frac{dE(\tilde{c})(s,t)}{ds}|_{s=0} = \int_a^b Re\{G(\boldsymbol{T}(\dot{c}_s + \bar{\dot{c}}_s, \dot{c}_t), \bar{\dot{c}}_t) + \frac{d}{dt}G(\dot{c}_s, \bar{\dot{c}}_t) - G(\dot{c}_s, D_{\dot{c}_t}\bar{\dot{c}}_t)$$
$$+ \frac{d}{dt}G(\dot{c}_s, \bar{\dot{c}}_t) - G(\dot{c}_s, D_{\bar{\dot{c}}_t}\bar{\dot{c}}_t)\}dt$$

for $s = 0$. But $\dot{c}_t$ and $\dot{c}_s$ are $\dot{\tilde{c}}_t$ and $\dot{\tilde{c}}_s$ for $s = 0$, and the variation is with fixed endpoints, $\dot{c}_t = V$ and $V(a) = V(b) = 0$, so we obtain the extremal values of the energy

$$\frac{1}{2}\frac{dE(\tilde{c})(s,t)}{ds}|_{s=0} = \int_a^b Re\{G(\boldsymbol{T}(\dot{c}_s + \overline{\dot{c}_s}, \dot{c}_t), \overline{\dot{c}_t}) - G(\dot{c}_s, D_{\dot{c}_{t+\overline{\dot{c}_t}}}\overline{\dot{c}_t}) = 0.$$

The extremal values of the first variation of the energy are named *the geodesic curves* of the metric *G* which are depending on the complex *N* –linear connection *D.*

From the above formula, taking into account that $\frac{dc}{dt} = \dot{c}_t + \overline{\dot{c}}_t$, we infer

Theorem 2.1. *The geodesic curves of the metric G verify the equations* $G\left(\dot{c}_s, D_{\frac{dc}{dt}}\overline{\dot{c}_t}\right) = G\left(\mathbf{T}\left(\frac{dc}{ds}, \dot{c}_t\right)\overline{\dot{c}_t}\right)$.

By restraining the calculations to *h*-, v_1-, v_2- curves *c*, we obtain the equations for the *h*-, v_1-, v_2- geodesics.

By a straightforward calculation $\mathbf{T}(\dot{c}_s + \overline{\dot{c}_s}, \dot{c}_t) = \mathbf{T}(\overset{(\alpha)}{\widetilde{U^j}}\, \delta_{\alpha j} + \overset{(\alpha)}{\widetilde{\overline{U}^j}}\, \delta_{\alpha \bar{j}}, \overset{(\beta)}{\widetilde{V^j}}\, \delta_{\beta j})$**,** with sum after $\alpha, \beta = 0{,}1{,}2$, we have the torsions $\mathbf{T}(\dot{c}_s + \overline{\dot{c}_s}, \dot{c}_t)$, taking into account the brackets obtained in [6].

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