JENSEN RELATED INEQUALITIES FOR CONVEXIFIABLE FUNCTIONS

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Abstract

In this paper we develop new Jensen like inequalities for nonconvex functions, like convexifiable functions. More precisely we refine the Jensen inequality for convexifiable functions and we conclude with a strong result concerning the interval of definition of convexifiable function on real achsis. More details on this work are provided in (Popescu, 2011, p. 120).

Key Words: *Convexity, Convexifiable functions, Jensen inequality, Refinements*

JEL Classification: C02

1. INTRODUCTION

Jensen's inequality is probably the most well-known inequality in economics. In microeconomics: E[U(w)] < U(E[w]) if U''(w) < 0 - For a risk averse agent, the expected utility of wealth is less than the utility of expected wealth. In international Macroeconomics/Finance: E[1/S] > 1/E[S] - The expectation of the reciprocal of an exchange rate is greater than the reciprocal of the expectation of the exchange rate. (Stastny, Online)

Jensen's inequality can also be used to bound different expressions in techniques, like the reduction of the exposure index induced by a network and many others quantities.

As an extension to convex functions we present the convexifiable functions, introduced in (Zlobec, 2006, p. 251-262) and in (Zlobec, 2004, p. 119-124), as follows

Definition 1 *Given a continuous function $f:R^n \to R$ defined on a convex set C and considering the function $\phi:R^n \to R$ defined by $\varphi(X,\alpha) = f(X) - \frac{1}{2}\alpha X^T X$, where X^T is the transposed of X, if $\phi(X,\alpha)$ is a convex function on C for some $\alpha=\alpha^*$, then $\phi(X,\alpha)$ is a convexification of f and α^* is its convexifier on C. Function f is convexifiable if it has a convexification.*

We continue by presenting the Jensen inequality for convexifiable functions on the real achsis (see Theorem 3 in (Zlobec, 2004, p. 119-124)), as follows

Theorem 1 *Consider a convexifiable scalar function $f:R \to R$ on a nontrivial compact interval [a,b] and its convexifier α. Then*

$$\frac{1}{n}\sum_{i=1}^{n} f(x_i) - f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) \geq \frac{\alpha}{2}\left(\frac{1}{n}\sum_{i=1}^{n} x_i^2 - \left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)^2\right),$$

for every set of n points $x_i \in [a,b], i = 1,\dots,n$.

In order to simplify the writing we denote by

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$$R(\alpha, n, x_1, x_2, ..., x_n) = \frac{\alpha}{2}\left(\frac{1}{n}\sum_{i=1}^{n} x_i^2 - \left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)^2\right)$$

and we observe that $R(\alpha, n, x_1, x_2, \ldots, x_n)$ has the same sign as α, because the function x^2 is a convex function.

2. JENSEN INEQUALITY FOR STRONGLY CONVEXIFIABLE FUNCTIONS WITH PARAMETER β

As a first refinement of the Jensen inequality for convexifiable functions we consider the concept of strongly convexifiable function, with parameter β>0, defined in the following

Definition 2 *Given a continuous function $f:R^n \to R$ defined on a convex set C and considering the function $\phi:R^n \to R$ defined by $\varphi(X,\alpha) = f(X) - \frac{1}{2}\alpha X^T X$, where X^T is the transposed of X, if $\phi(X,\alpha)$ is a strongly convex function, with parameter β>0, on C for some $\alpha=\alpha^*$, then $\phi(X,\alpha)$ is a strongly convexification, with parameter β>0, of f, and α^* is its convexifier on C. Function f is strongly convexifiable if it has a strongly convexification.*

The Jensen inequality for strongly convexifiable funtctions with parameter β>0 is presented into the following

Theorem 2 *Consider a strongly convexifiable scalar function, with parameter β>0, f:R→R on a nontrivial compact interval [a,b] and its convexifier α. Then*

$$\frac{1}{n}\sum_{i=1}^{n} f(x_i) - f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) \geq R(\alpha + \beta, n, x_1, x_2, ..., x_n),$$

for every set of n points $x_i \in [a,b], i = 1,\ldots,n$.

3. REFINEMENTS OF JENSEN INEQUALITY FOR CONVEXIFIABLE FUNCTIONS

We begin with a refinement of classical Jensen inequality (Jensen, 1906, p. 175-193), as follows

Theorem 3 *Considering a convex function, f:R→R, on a nontrivial compact interval [a,b], then*

$$\begin{aligned}\frac{1}{n}\sum_{i=1}^{n} f(x_i) &\geq \frac{1}{n}\left[n_1 f\left(\frac{1}{n_1}\sum_{i=1}^{n_1} x_i\right) + n_2 f\left(\frac{1}{n_2}\sum_{i=n_1+1}^{n_1+n_2} x_i\right) + \ldots\right.\\ &+ \left.\left(n - \sum_{k=1}^{m} n_k\right) f\left(\frac{1}{n - \sum_{k=1}^{m} n_k}\sum_{i=\sum_{k=1}^{m} n_k+1}^{n} x_i\right)\right]\\ &\geq f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)\end{aligned}$$

where $n_k \geq 1 (1 \leq k \leq m)$ are $m (m<n)$ positive integers with $\sum_{k=1}^{m} n_k < n$ and $x_i \in [a,b], i = 1,\ldots,n$.

In order to simplify the following Theorem we present an useful result, as follows

Lemma 1 *The following identity is true,*

$$R(\alpha,n,X_1,X_2,...,X_n)+\frac{1}{n}\left[n_1R(\alpha,n_1,x_1,...x_{n_1})+n_2R(\alpha,n_2,x_{n_1+1},...,x_{n_1+n_2})+...\right.$$
$$\left.+\left(n-\sum_{k=1}^{m}n_k\right)R\left(\alpha,n-\sum_{k=1}^{m}n_k,x_{\sum_{k=1}^{m}n_k+1},...,x_n\right)\right]=R(\alpha,n,x_1,...,x_n)$$

for every set of n points $x_i\in[a,b], i=1,\dots,n$ *where* $n_k\geq 1 (1\leq k\leq m)$ *are* $m(m<n)$ *positive integers with* $\sum_{k=1}^{m} n_k<n$ *and where*

$$X_1=X_2=...=X_{n_1}=\frac{\sum_{i=1}^{n_1}x_i}{n_1},\ X_{n_1+1}=X_{n_1+2}=...=X_{n_1+n_2}=\frac{\sum_{i=n_1+1}^{n_1+n_2}x_i}{n_2},...,$$
$$X_{\sum_{k=1}^{m}n_k+1}=X_{\sum_{k=1}^{m}n_k+2}=...=X_n=\frac{\sum_{i=\sum_{k=1}^{m}n_k+1}^{n}x_i}{n-\sum_{k=1}^{m}n_k}.$$

We continue presenting the equivalent form of the result from Theorem 3 for convexifiable functions, which consists in a refinement of Jensen inequality for convexifiable functions, as follows

Theorem 4 *Consider a convexifiable function f:R→R on a nontrivial compact interval [a,b] and its convexifier α. Then*

$$\begin{aligned}\frac{1}{n}\sum_{i=1}^{n}f(x_i) &- f\left(\frac{1}{n}\sum_{i=1}^{n}x_i\right)\geq\frac{1}{n}\left[n_1\left(R(\alpha,n_1,x_1,...x_{n_1})+f\left(\frac{1}{n_1}\sum_{i=1}^{n_1}x_i\right)\right)\right.\\
&+ n_2\left(R(\alpha,n_2,x_{n_1+1},...,x_{n_1+n_2})+f\left(\frac{1}{n_2}\sum_{i=n_1+1}^{n_1+n_2}x_i\right)\right)+...\\
&+ \left(n-\sum_{k=1}^{m}n_k\right)\left(R(\alpha,n-\sum_{k=1}^{m}n_k,x_{\sum_{k=1}^{m}n_k+1},...,x_n)\right.\\
&+ \left.\left.f\left(\frac{1}{n-\sum_{k=1}^{m}n_k}\sum_{i=\sum_{k=1}^{m}n_k+1}^{n}x_i\right)\right)\right]-f\left(\frac{1}{n}\sum_{i=1}^{n}x_i\right)\\
&\geq R(\alpha,n,x_1,...,x_n)\end{aligned}$$

for every set of n points $x_i\in[a,b], i=1,\dots,n$ *where* $n_k\geq 1 (1\leq k\leq m)$ *are* $m(m<n)$ *positive integers with* $\sum_{k=1}^{m} n_k<n$.

Trying to obtain a strong refinement of the previous result, we considered the case when the convexifiable function f defined on a nontrivial compact interval [a,b] is not concave on [a,b]. This means that there exist p intervals $[a_i,b_i]\subseteq[a,b](1\leq i\leq p)$, with $a_{i+1}>b_i(1\leq i\leq p-1)$, such that the function f is convex on each $[a_i,b_i](1\leq i\leq p)$ interval. So for any set of n points $x_i\in[a,b], i=1,\dots,n$ if we divide the sum of the points into subsums of points, such that all the points from one subsum are in an interval $[a_i,b_i]$, where the function f is convex, we obtain the following result

Theorem 5 *Consider a convexifiable function f:R→R on a nontrivial compact interval* *[a,b] and its convexifier α. If there exist p intervals* $[a_i, b_i] \subseteq [a,b] (1 \le i \le p)$, *with* $a_{i+1} > b_i (1 \le i \le p-1)$, *such that the function f is convex on each* $[a_i, b_i] (1 \le i \le p)$ *interval, then*

$$\begin{aligned}
\frac{1}{n}\sum_{i=1}^{n} f(x_i) &- f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) \ge \frac{1}{n}\left[n_1 f\left(\frac{\sum_{i=1}^{n_1} x_i}{n_1}\right) + (n_2 - n_1) f\left(\frac{\sum_{i=n_1+1}^{n_2} x_i}{n_2 - n_1}\right)\right. \\
&+ \left.\dots + (n - n_p)\left(R(\alpha, n - n_p, x_{n_p+1}, \dots, x_n) + f\left(\frac{\sum_{i=n_p+1}^{n} x_i}{n - n_p}\right)\right)\right] \\
&- f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) \\
&\ge R(\alpha, n, X_1, \dots, X_n) + \frac{n - n_p}{n} R(\alpha, n - n_p, x_{n_p+1}, \dots, x_n) \\
&\ge R(\alpha, n, x_1, \dots, x_n)
\end{aligned}$$

for every set of n points $x_i \in [a,b], i = 1, \dots, n$ *where* $n > n_p \ge \dots \ge n_1 \ge 1$ *are positive integers, such that* $x_1, \dots x_{n_1} \in [a_1, b_1]$, $x_{n_1}, \dots, x_{n_2} \in [a_2, b_2], \dots, x_{n_{p-1}}, \dots x_{n_p} \in [a_p, b_p]$, *and where*

$$X_1 = X_2 = \dots = X_{n_1} = \frac{\sum_{i=1}^{n_1} x_i}{n_1},\ X_{n_1+1} = X_{n_1+2} = \dots = X_{n_2} = \frac{\sum_{i=n_1+1}^{n_2} x_i}{n_2 - n_1}, \dots,$$
$$X_{n_p+1} = X_{n_p+2} = \dots = X_n = \frac{\sum_{i=n_p+1}^{n} x_i}{n - n_p}.$$

4. ACKNOWLEDGEMENTS

This work was partially supported by the EC under grant agreement FP7-ICT-318273 LEXNET project.

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